EXTREMAL LENGTH AND HARMONIC FUNCTIONS
ON RIEMANN SURFACES

BY

CARL DAVID MINDA[1]

ABSTRACT. Expressions for several conformally invariant pseudometrics on a Riemann surface R are given in terms of three new forms of reduced extremal distance. The pseudometrics are defined by means of various subclasses of the set of all harmonic functions on R having finite Dirichlet integral. The reduced extremal distance between two points is defined on R, on the Alexandroff one-point compactification of R and on the Kerékjártó-Stoïlow compactification of R. These reduced extremal distances are computed in terms of harmonic functions having specified singularities and boundary behavior. The key to establishing this connection with harmonic functions is a general theorem dealing with extremal length on a compact bordered Riemann surface and its extensions to noncompact bordered surfaces. These results are used to obtain new tests for degeneracy in the classification theory of Riemann surfaces. Finally, some of the results are illustrated for a hyperbolic simply connected Riemann surface.

Introduction. Three new forms of reduced extremal distance are defined for a Riemann surface R. These are related to the concept of the same name defined by Ahlfors and Beurling [2] and to their notion of perimeter [1]. For example, the reduced extremal distance $\lambda(\zeta_1, \zeta_2)$ between two points ζ_1, ζ_2 on R is defined by removing disks of radii r_1, r_2 about ζ_1, ζ_2, respectively, considering the extremal length of the family of all curves on R joining the boundaries of these disks and then adding the factor $(1/2\pi) \log (r_1 r_2)$ in order to obtain a quantity with a finite limit $\lambda(\zeta_1, \zeta_2)$ as $r_1, r_2 \to 0$. $\lambda'(\zeta_1, \zeta_2)$ and $\lambda^*(\zeta_1, \zeta_2)$ are defined in a similar manner except that the curves between the disks are taken on the Alexandroff one-point compactification R' and the Kerékjártó-Stoïlow compactification R^*, respectively. These reduced extremal distances are computed in terms of harmonic functions having prescribed singularities and boundary behavior (Theorems 5, 6). The key to establishing this connection with harmonic functions is a general result (Theorem 1) dealing with extremal distance on compact bordered Riemann surfaces

and its extensions (Theorems 2, 3, 4) to noncompact bordered surfaces which generalize a theorem of Marden and Rodin [4].

Next, several conformally invariant pseudometrics are defined from classes of harmonic functions. Let $HD(R)$ be the class of all harmonic functions u on R having finite Dirichlet integral $D(u)$. A conformally invariant pseudometric $d_{HD}(\zeta_1, \zeta_2)$ is introduced on R by taking the supremum of $|u(\zeta_1) - u(\zeta_2)|$ over all $u \in HD(R)$ with $D(u) \leq 1$. Two other pseudometrics, d_{KD} and d_{HM}, are defined analogously by employing the subclasses $KD(R)$ and $HM(R)$ of $HD(R)$. $KD(R)$ is the subset of $u \in HD(R)$ such that the integral of $* du$ over every dividing cycle γ on R is 0. A cycle γ on R is called dividing if for every compact set $E \subset R$ there is a cycle in $R \sim E$ homologous to γ. $HM(R)$, the space of harmonic measures, is the orthogonal complement of $KD(R)$ in the Hilbert space $HD(R)$.

Then (Theorem 7) it is shown that $d_{HD} = \lambda - \lambda'$, $d_{KD} = \lambda - \lambda^*$ and $d_{HM} = \lambda^* - \lambda'$, which provides a geometric extremal length interpretation of the various pseudometrics. Finally, this result is applied to derive tests for degeneracy in the classification theory of Riemann surfaces. A Riemann surface R is of class O_{HD}, O_{KD}, O_{HM} if the class $HD(R)$, $KD(R)$, $HM(R)$, respectively, contains only constant functions. It is proved (Theorems 8, 9) that $R \in O_{HD}$ (O_{KD}, O_{HM}) if and only if the extremal distance between any two compact subsets of R is the same whether computed on R or $R'(R$ or R^*, R' or $R^*)$. The result for O_{KD} is due to Rodin [6].

Finally, several explicit computations of the reduced extremal distances and a conformally invariant metric are given for hyperbolic simply connected Riemann surfaces.

1. **Preliminaries.** For general results about Riemann surfaces see Ahlfors-Sario [3].

(1.1) At times we shall require the existence and perhaps uniqueness of a harmonic function having certain singularities and prescribed behavior in a neighborhood of the ideal boundary β of a Riemann surface R. This will usually follow from the main existence theorem of the theory of normal operators ([3, Theorem 3A, p. 154], [8, Theorem 2B, p. 41]). A knowledge of the particular normal operators L_0, $(I)L_1$ and $(Q)L_1$ is necessary [8, pp. 46–51]. These operators give precise meaning to the boundary behavior of a harmonic function on β. For example, if u is harmonic on R, then $u = L_0 u$ in a neighborhood of β is akin to having a vanishing normal derivative on the boundary; that is, $* du = 0$ along the boundary where $* du = - u_y dx + u_x dy$ is the star conjugate of the differential form $du = u_x dx + u_y dy$ on R. Likewise, $u = (I)L_1 u$ near β means that u is constant on β and $\int_\beta * du = 0$, while $u = (Q)L_1 u$ implies that u is constant on each contour of β and the integral of $* du$ over each contour of β vanishes. Of course, both of these statements only hold in an appropriate limiting sense on a general Riemann surface,

but they are to be taken literally on a compact bordered Riemann surface. Recall that the Dirichlet integral of a harmonic function u on a Riemann surface R is

$$D_R(u) = D(u) = \iint\limits_R (u_x^2 + u_y^2)\, dx\, dy.$$

The subscript R is frequently omitted when one fixed Riemann surface is under consideration.

(1.2) The notion of extremal length for a family of chains on a Riemann surface will be employed. For a treatment of this topic see Rodin-Sario [8, pp. 121–132] and Sario-Oikawa [9, Appendix I]. A curve on R is a continuous mapping $c : I \to R$ where I is an open, closed or half-open interval on the real line. The curve c is locally rectifiable if the restriction of c to any compact subinterval of I is rectifiable. A locally rectifiable chain on R is a countable formal sum $c = \Sigma\, n_j c_j$ where n_j is an integer and c_j is a locally rectifiable curve. Henceforth, all curves and chains are assumed to be locally rectifiable. Let $ds = \rho(z)|dz|$ be a linear density on R and let c be a locally rectifiable curve. $\int_c ds$ is defined to be the supremum of the integrals of ds over all compact subpaths of c. For a locally rectifiable chain $c = \Sigma\, n_j c_j$ set

$$\int_c ds = \sum |n_j| \int_{c_j} ds.$$

This integral is called the ds or ρ length of c and is denoted by $L(c, ds) = L(c, \rho)$. The ds area of R is

$$A(R, ds) = A(ds) = \iint\limits_R \rho^2(z)\, dx\, dy.$$

Given a family $\mathcal{F}$ of locally rectifiable chains on R and a linear density ds, set

$$L(\mathcal{F}, ds) = \inf_{c \in \mathcal{F}} L(c, ds).$$

With the convention that $0/0 = \infty/\infty = 0$, the quantity $\lambda(\mathcal{F}) = \sup\, (L^2(\mathcal{F}, ds)/A(R, ds))$ is called the extremal length of $\mathcal{F}$; the supremum is taken over all linear densities ds on R.

2. **Extremal distance on compact bordered surfaces.** Let $\overline{R}$ be a compact bordered Riemann surface with interior R and border $\partial \overline{R}$. Suppose that the contours of $\partial \overline{R}$ are partitioned into disjoint sets $\alpha_0, \alpha_1, \gamma_0, \gamma_1$ with α_0, α_1 nonempty. When γ_1 is nonvoid let the contours in γ_1 be divided into nonempty disjoint sets, say $\beta_1, \beta_2, \cdots, \beta_m$. Each β_j is called a part of γ_1. Thus, $\{\alpha_0, \alpha_1, \gamma_0, \beta_1, \cdots, \beta_m\}$ represents a partition of $\partial \overline{R}$. Let $\mathcal{F}$ be the class of finite chains in $\overline{R} \sim \gamma_0$ which join α_0 to α_1, possibly via some of the parts of γ_1. More precisely, a finite chain $\Sigma_{j=1}^k c_j$ belongs to $\mathcal{F}$ if and only if either $k = 1$ and c_1 is a curve in $\overline{R} \sim \gamma_0$ which connects α_0 and α_1 or $k \geq 2$ and the $c_j : [0, 1] \to \overline{R} \sim \gamma_0$ are

curves with $c_1(0) \in \alpha_0$, $c_k(1) \in \alpha_1$, and for $1 \leq j \leq k-1$ the points $c_j(1)$ and $c_{j+1}(0)$ belong to the same part of γ_1. If the points in each part of γ_1 were identified and $\overline{R}$ given the induced quotient topology, then each chain in $\mathcal{F}$ could be viewed as a continuous curve in this quotient space from α_0 to α_1. In case $\gamma_1 = \emptyset$ the extremal length $\lambda(\mathcal{F})$ is customarily called the extremal distance between α_0 and α_1. The same designation for $\lambda(\mathcal{F})$ will still be used when $\gamma_1 \neq \emptyset$.

The following theorem generalizes a result of Marden and Rodin ([4, Lemma III. 1.1, p. 250], [8, Theorem 1D, p. 124]); the proof is an adaptation of theirs. They partition γ_1 into its constituent contours; that is, each β_j is a single contour of γ_1. The extension allows γ_1 to be separated into sets of contours in an arbitrary manner.

Theorem 1. $\lambda(\mathcal{F}) = D_{\overline{R}}(u)^{-1}$ *where* u *is the harmonic function on* $\overline{R}$ *such that* $u = 0$ *on* α_0, $u = 1$ *on* α_1, $* du = 0$ *along* γ_0, u *is a constant* c_j, $0 \leq c_j \leq 1$, *on each* β_j $(1 \leq j \leq m)$ *and* $\int_{\beta_j} * du = 0$ *for all* j.

Proof. The existence and uniqueness of u follows from the construction of the normal operators L_0 and L_1 [8, pp. 46–48]. An appropriate direct sum of the two operators L_0 and L_1 is required to produce the desired boundary behavior on γ_0 and γ_1.

First we show that $\lambda(\mathcal{F}) \geq D(u)^{-1}$ by using the particular linear density $ds = |du + i * du| = |u_x - iu_y||dz|$. The ds-area of $\overline{R}$ is $A(ds) = D(u)$. Observe that $ds \geq |du|$ and that the variation of u along any chain in $\mathcal{F}$ is at least 1, the difference between the values of u on α_0 and α_1. Consequently, $L(\mathcal{F}, ds) \geq 1$ and

$$D_{\overline{R}}(u)^{-1} \leq L^2(\mathcal{F}, ds)A(ds)^{-1} \leq \lambda(\mathcal{F}).$$

In order to establish the reverse inequality we shall consider chains which are locally the level curves of a harmonic conjugate u^* of u; that is, curves along which $* du = 0$. We shall find a parametrization of these level curves such that all but finitely many belong to $\mathcal{F}$.

It is necessary to orient and parametrize the boundary curves of $\overline{R}$. Orient the contours of $\partial \overline{R}$ so that $\overline{R}$ lies to the left. All of the contours in α_j $(j = 0, 1)$ shall be parametrized by a single function $a_j(t)$ defined on $[0, 1)$ such that as t goes from 0 to 1 the point $a_j(t)$ traverses each curve in α_j precisely once in the positive direction. Recall that α_j is a finite union of closed analytic Jordan curves. If α_j contains n contours, divide $[0, 1)$ into n half-open subintervals each of length $1/n$. The points of subdivision introduced are $t_k = k/n$ $(0 \leq k \leq n)$. Let a_j restricted to the subinterval $[t_k, t_{k+1})$ be an injective parametrization of a single contour of α_j which agrees with the orientation. Thus $a_j(t)$, $t \in [0, 1)$, is a path which traverses α_j once in the positive direction. Likewise, parametrize each part β_j of γ_1 by $b_j(t)$, $t \in [0, 1)$ $(1 \leq j \leq m)$.

These fixed parametrizations are used in continuing a harmonic conjugate of u along a portion of $\partial \bar{R}$ in a consistent manner. We illustrate this for α_0. Suppose u^* is a local harmonic conjugate of u defined near $a_0(0)$. Through harmonic continuation the function u^* may be defined along the curve $a_0([0, t_1))$. The value $u^*(a_0(t_1^-))$ exists but need not be equal to $u^*(a_0(0))$. Extend u^* to the curve $a_0([t_1, t_2))$ as follows. First determine a conjugate function u_1^* for u in a neighborhood of $a_0(t_1)$ such that $u_1^*(a_0(t_1)) = u^*(a_0(t_1^-))$ and then continue u_1^* harmonically along $a_0([t_1, t_2))$. In this manner we extend u^* to all of α_0 so that $u^*(a_0(t))$ is a continuous function on $[0, 1)$; in short, u^* is continuous on α_0. Functions are to be continued around α_1 and the β_j in a similar fashion. The fact that the integral of $*du = du^*$ over β_j vanishes guarantees that u^* returns to its initial value after traversing β_j once. Explicitly, $u^*(b_j(t))$ may be extended to the real line as a continuous function having period 1.

The function $u^*(a_0(t))$ strictly decreases from 0 to $-D(u)$ on $[0, 1)$ provided u^* is normalized so that $u^*(a_0(0)) = 0$. Since $0 \leq u \leq 1$ on $\bar{R}$, in a relative neighborhood of each point of α_0 the holomorphic function $f = u + iu^*$ has nonnegative real part and is purely imagininary on α_0. Therefore in terms of any local coordinate z mapping such a neighborhood onto a relatively open subset of the upper half-plane, the function $f(z)$ has a purely imaginary derivative and $\text{Im}\{f'(z)\} < 0$ on α_0. In other words, $du^* = *du < 0$ along α_0 so that u^* is strictly decreasing. From

$$D_{\bar{R}}(u) = \int_{\partial \bar{R}} u * du = \int_{\alpha_1} * du = -\int_{\alpha_0} * du,$$

we see that u^* decreases from 0 to $-D(u)$.

Two complementary subsets, S and C, of the interval $(-D(u), 0]$ shall be defined. C will be finite and with each $s \in S$ we shall associate a chain in $\mathcal{F}$. For each $s \in (-D(u), 0]$ a single level curve $l_0(s)$ of u^* leaves α_0. Orient $l_0(s)$ so that u increases in the positive direction. $l_0(s)$ is a curve along which $*du = 0$ and is an analytic Jordan arc unless it passes through a critical point of u. In case the connected level curve $l_0(s)$ passes through a critical point of u in $\bar{R}$, place s in C. If $l_0(s)$ does not cross any critical points of u, then it ends on either α_1 or γ_1. When $l_0(s)$ terminates on α_1, put s in S and write $c(s)$ in place of $l_0(s)$. Note that $c(s) \in \mathcal{F}$. Otherwise, $l_0(s)$ stops on some $\beta_j \subset \gamma_1$. Let $\zeta(s)$ denote the endpoint of $l_0(s)$ on β_j; our assumption about $l_0(s)$ guarantees that $\zeta(s)$ is not a critical point of u. Now move along β_j in the positive direction following the path determined by b_j. The function u^* is initially increasing as we continue u^* from $\zeta(s)$ along β_j. Continue u^* along β_j until arriving at $\zeta'(s)$, the first point for which $u^*(\zeta'(s)) = u^*(\zeta(s))$. Such a point $\zeta'(s)$ exists because the integral of $*du$ over β_j is 0. For future reference we note that $\zeta'(s) = \zeta'(t)$ if and only if $\zeta(s) = \zeta(t)$. If $\zeta'(s)$ is a critical point of u, then put s in C. If not, then the level curve $l_1(s)$ of u^* which originates at $\zeta'(s)$ moves into R

when oriented in the direction of increasing values of u. In fact, $* du < 0$ along β_j near $\zeta'(s)$ since $* du > 0$ along β_j by $\zeta(s)$; this also shows that $\zeta'(s) \neq \zeta(s)$. Thus, $u + iu^*$ maps a relative neighborhood of $\zeta'(s)$ onto a relative open set in $\{z \colon \operatorname{Re}\{z\} \geq c_j\}$ where $u \equiv c_j$ on β_j. The level curve $l_1(s)$ is called a continuation of $l_0(s)$. If $l_1(s)$ crosses a critical point of u, then s goes in C. Otherwise, u is strictly increasing on $l_1(s)$ so that the connected level curve $l_1(s)$ ends on either α_1 or some β_k with $k \neq j$. In the former case, write $c(s) = l_0(s) + l_1(s)$ and put s in S. Again $c(s) \in \mathcal{F}$. In the latter we repeat the construction given above.

In this manner we see that each $s \in (-D(u), 0]$ is in either C or S. For every $s \in S$ there is a finite chain $c(s) = l_0(s) + \cdots + l_{j(s)}(s)$ of connected level curves of u^* obtained by successive continuation beginning with $l_0(s)$ and $c(s) \in \mathcal{F}$. For $s \neq t$ in S the chains $c(s)$ and $c(t)$ are disjoint. For if $l_j(s) \cap l_k(t) \neq \varnothing$, then $l_j(s) = l_k(t)$ because they are level curves for the same harmonic function and do not pass through any critical points. If $j = k = 0$, then $s = t$ is immediate. Otherwise, $j, k \geq 1$ and $l_j(s), l_k(t)$ have the same initial point on some part of γ_1 so that $l_{j-1}(s)$ and $l_{k-1}(t)$ must have the same terminal point. But then $l_{j-1}(s) = l_{k-1}(t)$. Continuing in this manner we find that $l_0(s) = l_0(t)$ and $s = t$. The set C is finite because u has only finitely many critical points on the compact surface $\overline{R}$ and through each such point pass only finitely many level curves of u^*.

Finally we establish the reverse inequality. Except at the finite number of critical points of u the function $u + iu^*$ can be used as a local parameter and $\overline{R}$ can be paved with little rectangles determined by the level curves of u and u^*. For any linear density $ds = \rho|dz|$ and $s \in S$,

$$\int_{c(s)} \rho \, |dz| = \int_{c(s)} \rho \, du \leq \left(\int_{c(s)} \rho^2 \, du \right)^{\!\!1/2}.$$

Here we have used the facts that $* du = 0$, $du > 0$ along $c(s)$ and that the integral of du over $c(s)$ is 1. Since $\{c(s) \colon s \in S\} \subset \mathcal{F}$,

$$L^2(\mathcal{F}, ds) \leq \left(\int_{c(s)} \rho \, du \right)^{\!2} \leq \int_{c(s)} \rho^2 \, du.$$

Integration over S results in

$$D_{\overline{R}}(u) L^2(\mathcal{F}, ds) \leq \int_{s \in S} \int_{c(s)} \rho^2 \, du \, du^* = A(ds),$$

so that $\lambda(\mathcal{F}) \leq D(u)^{-1}$, which completes the proof.

3. **Extensions to noncompact bordered surfaces.** Next we present three generalizations of Theorem 1 to the case in which $\overline{R}$ is a noncompact bordered Riemann surface having compact border $\partial \overline{R}$. In this section $\overline{R}$ is fixed and the contours of $\partial \overline{R}$ are partitioned into two nonempty disjoint sets, α_0 and α_1.

(3.1) The first generalization corresponds to Theorem 1 when $\gamma_1 = \emptyset$. In this situation we are interested in the extremal length of the family $\mathcal{F}$ of all curves in $\overline{R}$ which join α_0 to α_1; that is, the extremal distance of α_0 and α_1. The following result is known ([4, Theorem III. 3.1, p. 256], [8, Theorem 2A, p. 156]), but the proof is new and brief and is included for completeness.

Theorem 2. $\lambda(\mathcal{F}) = D_{\overline{R}}(u_0)^{-1}$ *where* u_0 *is the harmonic function on* $\overline{R}$ *such that* $u_0 = 0$ *on* α_0, $u_0 = 1$ *on* α_1 *and* $u_0 = L_0 u_0$ *in a neighborhood of the ideal boundary of* $\overline{R}$.

Proof. Let $\{\overline{R}_n\}_{n=1}^{\infty}$ be a regular exhaustion of $\overline{R}$; that is, $\{\overline{R}_n\}$ is a strictly increasing sequence of compact bordered Riemann surfaces such that $\overline{R} = \bigcup \overline{R}_n$ and $\partial \overline{R}_n \supset \partial \overline{R} = \alpha_0 \cup \alpha_1$ for all n. Set $\gamma_n = \partial \overline{R}_n \sim \partial \overline{R}$, then $\{\alpha_0, \alpha_1, \gamma_n\}$ is a partition of $\partial \overline{R}_n$ into disjoint sets of contours with α_0, α_1 nonempty. Let u_n be the harmonic function on $\overline{R}_n$ with the properties $u_n = 0$ on α_0, $u_n = 1$ on α_1 and $*du_n = 0$ along γ_n. If $\mathcal{F}_n$ is the family of all curves in $\overline{R}_n \sim \gamma_n$ joining α_0 to α_1, then Theorem 1 gives $\lambda(\mathcal{F}_n) = D(u_n)^{-1}$ where the Dirichlet integral is taken over $\overline{R}_n$. Now, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n and $\mathcal{F} = \bigcup \mathcal{F}_n$ so that $\lambda(\mathcal{F}) = \lim \lambda(\mathcal{F}_n)$ [9, p. 321]. The sequence $\{u_n\}$ is uniformly convergent on compact subsets of $\overline{R}$ to u_0 and $D(u_n)$ converges to $D(u_0)$ [8, Theorem 2B, p. 58]. Hence, $\lambda(\mathcal{F}) = D(u_0)^{-1}$.

(3.2) The next extension of Theorem 1 is the analog of the case in which $\gamma_0 = \emptyset$ and γ_1 is given the identity partition; in other words, all contours of γ_1 belong to the same part. Let $\overline{R}'$ be the Alexandroff one-point compactification of $\overline{R}$ and β the point added to $\overline{R}$ to obtain $\overline{R}'$. Given a sequence $\{\zeta_n\}$ of points in $\overline{R}$, write $\zeta_n \to \beta$ if for every compact set $E \subset \overline{R}$ there is an integer $N = N(E)$ with the property that $\zeta_n \notin E$ for all $n \geq N$. Similarly, if $l : [0, 1) \to \overline{R}$ is a curve, then $l(t) \to \beta$ as $t \to 1$ means that for each compact set $E \subset \overline{R}$ there is a $\delta > 0$ such that $l(t) \notin E$ for $1 - \delta < t < 1$. Say that $l(t)$ clusters on β as $t \to 1$ if there is a sequence $t_n \to 1$ with $l(t_n) \to \beta$. Let $\mathcal{F}'$ be the family of curves in $\overline{R}'$ connecting α_0 to α_1. $\lambda(\mathcal{F}')$ is the extremal length of the family of chains obtained by restricting each curve in $\mathcal{F}'$ to $\overline{R}$. Since $\mathcal{F}' \supset \mathcal{F}$, it follows that $\lambda(\mathcal{F}') \leq \lambda(\mathcal{F})$. Two related families $\mathcal{G}'$ and $\mathcal{H}'$ are needed. $\mathcal{G}'$ is the family of chains l such that either l is a curve in $\overline{R}$ joining α_0 to α_1 or $l = l_0 + l_1$ where $l_j : [0, 1) \to \overline{R}$ is a curve with $l_j(0) \in \alpha_j$ and $l_j(t)$ clusters on β as $t \to 1$ $(j = 0, 1)$. $\mathcal{H}'$ is the subfamily of $\mathcal{G}'$ consisting of the curves in $\overline{R}$ joining α_0 to α_1 together with the chains $l_0 + l_1$ such that $l_j(t) \to \beta$ as $t \to 1$. If we define $l_j(1) = \beta$ in this case, then l_j becomes a curve from α_j to β in $\overline{R}'$ and $l_0 + l_1$ may be viewed as a curve in $\overline{R}'$ joining α_0 to α_1. In this natural way we regard $\mathcal{H}'$ as a subset of $\mathcal{F}'$. Both $\mathcal{G}'$ and $\mathcal{H}'$ contain $\mathcal{F}$. The following result shows that $\mathcal{F}'$, $\mathcal{G}'$ and $\mathcal{H}'$ are equivalent families for extremal length problems.

Proposition 1. $\lambda(\mathcal{F}') = \lambda(\mathcal{G}') = \lambda(\mathcal{H}')$.

Proof. The inclusion $\mathcal{H}' \subset \mathcal{F}'$ implies $\lambda(\mathcal{H}') \geq \lambda(\mathcal{F}')$. Every curve in $\mathcal{F}'$ contains a chain in $\mathcal{H}'$ so that $\lambda(\mathcal{F}') \geq \lambda(\mathcal{H}')$ [9, p. 321]. Therefore, $\lambda(\mathcal{F}') = \lambda(\mathcal{H}')$.

In order to establish $\lambda(\mathcal{H}') = \lambda(\mathcal{G}')$ it is sufficient to show that $\lambda(\mathcal{G}' \sim \mathcal{H}') = \infty$ [8, Corollary 2A, p. 128]. The family $\mathcal{G}' \sim \mathcal{H}'$ has infinite extremal length because every chain in this family must oscillate. Let $\{\overline{R}_n\}$ be a regular exhaustion of $\overline{R}$ and set $\gamma_n = \partial \overline{R}_n \sim \partial \overline{R}$. About each contour of every γ_n place an annulus which is contained in $\overline{R}_{n+1} \sim \overline{R}_{n-1}$ such that all of the annuli are pairwise disjoint. We shall prove that every chain in $\mathcal{G}' \sim \mathcal{H}'$ crosses some annulus infinitely often. Then $\lambda(\mathcal{G}' \sim \mathcal{H}') = \infty$ follows because the family of chains each of which crosses at least one of the annuli infinitely often has infinite extremal length.

If $l \in \mathcal{G}' \sim \mathcal{H}'$, then $l = l_0 + l_1$ where both l_0 and l_1 cluster on β but one of them does not actually tend to β since $l \notin \mathcal{H}'$. We consider the case in which l_0 does not approach β as $t \to 1$. Then there is a sequence $\{s_n\}$ tending to 1 and an $\overline{R}_N$ such that $l_0(s_n) \in \overline{R}_N$ for all n. Since l_0 clusters on β, there is also a sequence $\{t_n\}$ for which $t_n \to 1$ and $l_0(t_n) \notin \overline{R}_{N+2}$ for all n. By extracting subsequences if necessary we may assume that $s_n < t_n < s_{n+1}$ for all n. In the interval $[s_n, t_n]$ the curve l_0 crosses at least one of the annular regions about the contours of γ_{N+1}. As there are but finitely many contours in γ_{N+1}, the curve l_0 must cross one of the annuli infinitely often.

This proposition simplifies the proof of next theorem.

Theorem 3. $\lambda(\mathcal{F}') = D_{\overline{R}}(u_I)^{-1}$ *where* u_I *is the harmonic function on* $\overline{R}$ *such that* $u_I = 0$ *on* α_0, $u_I = 1$ *on* α_1 *and* $u_I = (I)L_1 u_I$ *in a neighborhood of the ideal boundary of* $\overline{R}$.

Proof. The proof makes use of an extremal length technique ascribed to Beurling and developed by Wolontis [12, p. 591]. We employ the method in a manner related to the work of Strebel [10, pp. 10–11], Marden and Rodin [4, pp. 252–256] and Suita [11, pp. 239–242]. The proof is divided into two parts: first we show that $\lambda(\mathcal{G}'_n) \to \lambda(\mathcal{G}')$ where $\mathcal{G}'_n$ is a curve family associated with $\overline{R}_n$, part of a regular exhaustion of $\overline{R}$, and then we use this result to show that $\lambda(\mathcal{G}') = D(u_I)^{-1}$.

Let $\{\overline{R}_n\}$ be a regular exhaustion of $\overline{R}$ and set $\gamma_n = \partial \overline{R}_n \sim \partial \overline{R}$. Let $\mathcal{G}'_n$ be the collection of all chains l such that either l is a curve in $\overline{R}_n \sim \gamma_n$ joining α_0 to α_1 or $l = l_0 + l_1$ where l_j is a curve in $\overline{R}_n$ connecting α_j to γ_n $(j = 0, 1)$. Because every chain in $\mathcal{G}'_{n+1}$ contains a chain in $\mathcal{G}'_n$, $\lambda(\mathcal{G}'_n) \leq \lambda(\mathcal{G}'_{n+1})$ for all n [9, p. 321], and hence $\lim \lambda(\mathcal{G}'_n)$ exists. Also, every chain in $\mathcal{G}'$ contains a chain in $\mathcal{G}'_n$ so that $\lim \lambda(\mathcal{G}'_n) \leq \lambda(\mathcal{G}')$. In order to prove that $\lim \lambda(\mathcal{G}'_n) = \lambda(\mathcal{G}')$, it remains to show that $\lambda(\mathcal{G}') \leq \lim \lambda(\mathcal{G}'_n)$. For this we may assume that $\lambda(\mathcal{G}') > 0$ and $\lim \lambda(\mathcal{G}'_n) < \infty$. Let $M > 0$ be any number strictly less than $\lambda(\mathcal{G}')$, then it suffices to demonstrate that $M \leq \lim \lambda(\mathcal{G}'_n)$. Now, there is a density ds with $M < L^2(\mathcal{G}', ds)A(R, ds)^{-1}$. Since $\lambda(\mathcal{G}') = \lambda(\mathcal{F}') \leq \lambda(\mathcal{F}) < \infty$, we may also assume that

$A(\bar{R}, ds) > 0$. Let $a_n = L(\mathcal{G}'_n, ds)$, then $\{a_n\}$ is an increasing sequence. Moreover,

$$a_n^2 \leq A(\bar{R}_n, ds)\lambda(\mathcal{G}'_n) \leq A(\bar{R}, ds)\lambda(\mathcal{G}'_n) \leq A(\bar{R}, ds) \lim \lambda(\mathcal{G}'_n),$$

so that $a = \lim a_n$ is finite and $a^2 \leq A(\bar{R}, ds) \lim \lambda(\mathcal{G}'_n)$. Let $\epsilon > 0$ be given. We shall prove there is an $l \in \mathcal{G}'$ such that $L(l, ds) < a + 5\epsilon$. Then $L(\mathcal{G}', ds) < a + 5\epsilon$ and $M < (a + 5\epsilon)^2 A(\bar{R}, ds)^{-1}$. Consequently, $M \leq a^2 A(\bar{R}, ds)^{-1} \leq \lim \lambda(\mathcal{G}'_n)$ which is the desired result.

Establishing the existence of l is a bit involved. For every n determine $l_n \in \mathcal{G}'_n$ with $L(l_n, ds) < a + \epsilon$. If for any value of n the chain l_n is actually a curve in $\bar{R}_n \sim \gamma_n$ joining α_0 to α_1, then we simply take $l = l_n$. Otherwise for every n, $l_n = l_{0n} + l_{1n}$ where l_{jn} is a curve in $\bar{R}_n$ from α_j to γ_n ($j = 0, 1$). Let $\zeta_{jn}(k)$ denote the first point of intersection of l_{jn} with γ_k for $k \leq n$. Initially we pass to a subsequence of $\{l_{jn}\}$ such that $\zeta_{jn}(1)$ converges to a point $\zeta_j(1) \in \gamma_1$ as $n \to \infty$. This is possible because γ_1 is compact. We continue to denote this subsequence by $\{l_{jn}\}$. From this new sequence we again extract a subsequence, this time so that the points $\zeta_{jn}(2)$ converge to $\zeta_j(2) \in \gamma_2$. Continue in this manner. The diagonal sequence, which is still designated as $\{l_{jn}\}$, has the property that $\zeta_{jn}(k) \to_n \zeta_j(k) \in \gamma_k$ for every k. Observe that in passing to a subsequence of $\{l_{jn}\}$ we have also gone to a subsequence of $\{\bar{R}_n\}$.

Let $V_j(k)$ be a parametric disk centered at $\zeta_j(k)$ such that the disks are all pairwise disjoint. Since the extremal length of the family of all Jordan curves in $V_j(k) \sim \{\zeta_j(k)\}$ which contain $\zeta_j(k)$ in their interior is 0 [9, p. 325], there is a Jordan curve $c_j(k)$ in $V_j(k) \sim \{\zeta_j(k)\}$ enclosing $\zeta_j(k)$ with $L(c_j(k), ds) < \epsilon 2^{-k}$.

Let $\mathfrak{D}_j(k)$ denote the family of all curves on $\bar{R}$ joining $c_j(k-1)$ to $c_j(k)$ for $k \geq 2$ and let $\mathfrak{D}_j(1)$ be the collection of all curves on $\bar{R}$ connecting α_j to $c_j(1)$ ($j = 0, 1$). Then

$$\sum_{k=1}^{\infty} L(\mathfrak{D}_j(k), ds) \leq \liminf_{n \to \infty} L(l_{jn}, ds).$$

For each integer $k \geq 1$ let $\delta_j(k) \in \mathfrak{D}_j(k)$ be such that $L(\delta_j(k), ds) \leq L(\mathfrak{D}_j(k), ds) + \epsilon 2^{-k}$. Let $\bar{c}_j(k)$ be a subarc of $c_j(k)$ which joins the end of $\delta_j(k)$ to the initial point of $\delta_j(k+1)$. Set

$$l_j = \sum_{k=1}^{\infty} [\delta_j(k) + \bar{c}_j(k)].$$

If l_j is parametrized in an appropriate manner, then $l = l_0 + l_1 \in \mathcal{G}'$. Also,

$$L(l_j, ds) = \sum_{k=1}^{\infty} [L(\delta_j(k), ds) + L(\bar{c}_j(k), ds)]$$

$$\leq 2\epsilon + \liminf_{n \to \infty} L(l_{jn}, ds).$$

From this inequality it follows that

$$L(l, ds) \leq 4\epsilon + \liminf_{n \to \infty} L(l_n, ds) = a + 5\epsilon,$$

so that $l \in \mathcal{G}'$ is an appropriate choice.

So far we have shown that $\lim \lambda(\mathcal{G}'_n) = \lambda(\mathcal{G}')$. Employing this result we may complete the proof. $\{\alpha_0, \alpha_1, \gamma_n\}$ is a partition of $\partial \overline{R}_n$ with α_0, α_1 nonempty. Let γ_n be given the identity partition in which all contours belong to the same part. Take $\mathcal{F}_n$ to be the family of all finite chains in $\overline{R}_n$ which join α_0 to α_1, possibly via γ_n. Explicitly, $l \in \mathcal{F}$ if and only if either l is a curve in $\overline{R} \sim \gamma_n$ from α_0 to α_1 or $l = \Sigma_{j=0}^m l_j$ where $m \geq 1$, l_j is a curve connecting α_j to γ_n $(j = 0, 1)$ and l_j is a curve joining any two contours in γ_n $(2 \leq j \leq m)$. By Theorem 1, $\lambda(\mathcal{F}_n) = D(u_n)^{-1}$ where u_n is harmonic on $\overline{R}_n$, $u_n = 0$ on α_0, $u_n = 1$ on α_1, u_n is a constant c on γ_n, $0 \leq c \leq 1$, and $\int_{\gamma_n} *du = 0$. Evidently, $\mathcal{G}'_n \subset \mathcal{F}_n$ so that $\lambda(\mathcal{G}'_n) \geq \lambda(\mathcal{F}_n)$. But each chain in $\mathcal{F}_n$ contains a chain in $\mathcal{G}'_n$ so that $\lambda(\mathcal{G}'_n) \leq \lambda(\mathcal{F}_n)$. Therefore, $\lambda(\mathcal{G}'_n) = \lambda(\mathcal{F}_n)$. The sequence $\{u_n\}$ converges uniformly to u_I on compact subsets of $\overline{R}$, and $D(u_n)$ tends to $D(u_I)$ [8, Theorem 2A, p. 58]. Hence, $\lambda(\mathcal{F}') = \lambda(\mathcal{G}') = D(u_I)^{-1}$.

(3.3) The third generalization involves the Kerékjártó-Stoïlow compactification $\overline{R}^*$ of $\overline{R}$; as is customary we view $\overline{R}$ as a subset of $\overline{R}^*$. Let $\mathcal{F}^*$ be the family of all curves in $\overline{R}^*$ joining α_0 to α_1. $\lambda(\mathcal{F}^*)$ is the extremal length of the family of chains obtained by restricting each curve in $\mathcal{F}^*$ to $\overline{R}$. Clearly, $\mathcal{F} \subset \mathcal{F}^* \subset \mathcal{F}'$ which gives $\lambda(\mathcal{F}') \leq \lambda(\mathcal{F}^*) \leq \lambda(\mathcal{F})$. This situation is analogous to Theorem 1 when $\gamma_0 = \emptyset$ and γ_1 is given the canonical partition in which each contour in γ_1 constitutes a part of γ_1. The following result is a special case of a theorem of Marden and Rodin ([4, Theorem III. 3.1, p. 256], [8, Theorem 2A, p. 156]); it is given here for future reference. They obtain it from their version of Theorem 1.

Theorem 4. $\lambda(\mathcal{F}^*) = D_{\overline{R}}(u_Q)^{-1}$ where u_Q is the harmonic function on $\overline{R}$ such that $u_Q = 0$ on α_0, $u_Q = 1$ on α_1 and $u_Q = (Q)L_1 u_Q$ in a neighborhood of the ideal boundary of $\overline{R}$.

4. **Reduced extremal distances.** Let R be a Riemann surface and ζ_1, ζ_2 two distinct points on R. We shall define three reduced extremal distances between ζ_1 and ζ_2: one on R; another on R', the Alexandroff one-point compactification of R; and a third on R^*, the Kerékjártó-Stoïlow compactification of R. The following notation will be standard in this section. Suppose that z_j is a local coordinate defined in a neighborhood N_j of ζ_j such that $z_j(\zeta_j) = 0$ $(j = 1, 2)$ and $N_1 \cap N_2 = \emptyset$. For all sufficiently small positive values of r the parametric disk $D_j(r) = \{\zeta \in R: |z_j(\zeta)| < r\}$ and its boundary $\alpha_j(r) = \{\zeta \in R: |z_j(\zeta)| = r\}$ are both subsets of N_j.

(4.1) Let $\mathcal{F}(r_1, r_2)$ designate the class of all curves on $R \sim [D_1(r_1) \cup D_2(r_2)]$

$= \overline{R}(r_1, r_2)$ which connect $\alpha_1(r_1)$ and $\alpha_2(r_2)$. The extremal length of the curve family $\mathcal{F}(r_1, r_2)$ is the extremal distance between the contours $\alpha_1(r_1)$ and $\alpha_2(r_2)$ on the bordered Riemann surface $\overline{R}(r_1, r_2)$.

Proposition 2. $\lambda(\mathcal{F}(r_1, r_2)) + (1/2\pi) \log(r_1 r_2)$ *increases if either* r_1 *or* r_2 *decreases.*

Proof. Suppose that $0 < r_j \le R_j$ and let $\mathcal{C}(r_j, R_j)$ be the family of all curves in the annular region $D_j(R_j) \sim \text{Cl}(D_j(r_j))$ which join the inner boundary contour $\alpha_j(r_j)$ to the outer curve $\alpha_j(R_j)$. Every curve in $\mathcal{F}(r_1, r_2)$ contains a curve from each of the families $\mathcal{C}(r_1, R_1)$, $\mathcal{F}(R_1, R_2)$ and $\mathcal{C}(r_2, R_2)$ and the supports of these three curve families are pairwise disjoint Borel sets. Hence [9, p. 321],

$$\lambda \mathcal{F}(r_1, r_2)) \ge \lambda(\mathcal{C}(r_1, R_1)) + \lambda(\mathcal{F}(R_1, R_2)) + \lambda(\mathcal{C}(r_2, R_2))$$

$$= \frac{1}{2\pi} \log \frac{R_1}{r_1} + \lambda(\mathcal{F}(R_1, R_2)) + \frac{1}{2\pi} \log \frac{R_2}{r_2} ,$$

which establishes the proposition.

The preceding result guarantees that $\lambda(\mathcal{F}(r_1, r_2)) + (1/2\pi) \log(r_1 r_2)$ approaches a limit (possibly $+\infty$) as $(r_1, r_2) \to (0, 0)$; the limit is the supremum taken over r_1, r_2 positive and sufficiently small. The number

$$\lambda(\zeta_1, \zeta_2) = \lim_{(r_1, r_2) \to (0,0)} [\lambda(\mathcal{F}(r_1, r_2)) + (1/2\pi) \log(r_1 r_2)]$$

is called the *reduced extremal distance* between ζ_1 and ζ_2. For convenience set $\lambda(\zeta_1, \zeta_2) = 0$ if $\zeta_1 = \zeta_2$. The value of $\lambda(\zeta_1, \zeta_2)$ depends upon the choice of the local parameter z_j at ζ_j. This dependence will be made explicit later. This concept of reduced extremal distance between two points is related to, but distinct from, the reduced extremal distance between a point and a compact set as defined by Ahlfors and Beurling ([2, p. 244], [5, pp. 238–241]), as well as their definition of the perimeter of a compact set relative to a point [1, pp. 115–116].

(4.2) Let $p_0(\zeta) = p_0(\zeta; \zeta_1, \zeta_2)$ be the L_0-principal function having a negative logarithmic singularity at ζ_1 and a positive logarithmic singularity at ζ_2. Thus, p_0 is harmonic on $R \sim \{\zeta_1, \zeta_2\}$, $p_0 = L_0 p_0$ in a neighborhood of the ideal boundary of R and $p_0(z_j) + (-1)^j \log|z_j|$ has a removable singularity at $z_j = 0$ ($j = 1, 2$). Let $p_0 \langle \zeta_j \rangle$ denote the value of the latter harmonic function at $z_j = 0$; in other words,

$$p_0 \langle \zeta_j \rangle = \lim_{z_j \to 0} [p_0(z_j) + (-1)^j \log|z_j|] \qquad (j = 1, 2).$$

The value of $p_0 \langle \zeta_j \rangle$ is defined with respect to the fixed local coordinate z_j. In general, the value of $p_0 \langle \zeta_j \rangle$ would be different if another local coordinate were used.

Theorem 5. $\lambda(\zeta_1, \zeta_2) = (1/2\pi)(p_0\langle\zeta_2\rangle - p_0\langle\zeta_1\rangle)$.

Proof. First we show that the right-hand side is an upper bound for $\lambda(\zeta_1, \zeta_2)$ by making use of a Riemann surface $\Omega(r)$ containing $\overline{R}(r, r)$ on which it is easier to compute the relevant extremal lengths. A similar type of argument is used in [5, Theorem 2.94, p. 243]. For small positive values of r let

$$m_1(r) = \min_{\zeta \in a_1(r)} p_0(\zeta), \qquad m_2(r) = \max_{\zeta \in a_2(r)} p_0(\zeta).$$

When r is sufficiently small, $m_1(r) < 0 < m_2(r)$ and $c_j(r) = \{\zeta : p_0(\zeta) = m_j(r)\}$ ($j = 1, 2$) is an analytic Jordan curve. Orient $c_j(r)$ so that the component of the complement of $c_j(r)$ which contains ζ_j lies to the left; in other words, $c_j(r)$ is oriented positively with respect to the point ζ_j. Notice that $c_j(r) \subset \text{Cl}(D_j(r))$. Set $\Omega(r) = \{\zeta : m_1(r) \leq p_0(\zeta) \leq m_2(r)\}$. $\Omega(r)$ is a bordered Riemann surface having compact border $c_1(r) \cup c_2(r)$ and $\Omega(r) \supset \overline{R}(r, r)$.

Let $\lambda(c_1(r), c_2(r))$ denote the extremal distance of $c_1(r)$ and $c_2(r)$ on the surface $\Omega(r)$; that is, the extremal length of the family of all curves on $\Omega(r)$ which join $c_1(r)$ to $c_2(r)$. Because every such curve contains a subcurve belonging to $\mathcal{F}(r, r)$, it follows that $\lambda(\mathcal{F}(r, r)) \leq \lambda(c_1(r), c_2(r))$ [9, p. 321]. It is a simple matter to compute $\lambda(c_1(r), c_2(r))$ in terms of the function p_0. The function $u(\zeta) = [m_2(r) - p_0(\zeta)][m_2(r) - m_1(r)]^{-1}$ is harmonic on $\Omega(r)$, 0 on $c_2(r)$, 1 on $c_1(r)$ and $u = L_0 u$ in a neighborhood of the ideal boundary of $\Omega(r)$. By Theorem 2 the extremal length $\lambda(c_1(r), c_2(r))$ is equal to the reciprocal of the Dirichlet integral $D(u)$ of u taken over $\Omega(r)$. An application of Stokes' theorem results in

$$[m_2(r) - m_1(r)]^2 D_{\Omega(r)}(u) = D_{\Omega(r)}(p_0)$$

$$= -\left[m_1(r) \int_{c_1(r)} {}^*dp_0 + m_2(r) \int_{c_2(r)} {}^*dp_0\right] = -2\pi[m_1(r) - m_2(r)].$$

In applying Stokes' theorem no term arises which corresponds to the ideal boundary since p_0 has L_0-behavior near the ideal boundary. (For future reference observe that $(I)L_1$ or $(Q)L_1$-behavior in a neighborhood of the ideal boundary yields the same conclusion.) Consequently, $\lambda(c_1(r), c_2(r)) = (1/2\pi)\lfloor m_2(r) - m_1(r)\rfloor$. From

$$\lim_{r \to 0} [m_2(r) - m_1(r) + \log(r^2)] = p_0\langle\zeta_2\rangle - p_0\langle\zeta_1\rangle,$$

it is clear that $\lambda(\zeta_1, \zeta_2) \leq (1/2\pi)[p_0\langle\zeta_2\rangle - p_0\langle\zeta_1\rangle]$. The opposite inequality is obtained by a similar procedure but using an inner approximation to $\overline{R}(r, r)$ instead.

(4.3) Theorem 5 provides an easy means to determine the dependence of $\lambda(\zeta_1, \zeta_2)$ on the choice of local parameters at ζ_1 and ζ_2. Suppose that w_j is a local coordinate defined in a neighborhood M_j of ζ_j with $w_j(\zeta_j) = 0$. Let the

reduced extremal distance between ζ_1 and ζ_2 relative to these coordinates be denoted by $\lambda_w(\zeta_1, \zeta_2)$ and let $\lambda_z(\zeta_1, \zeta_2)$ designate the reduced extremal distance when computed with respect to the coordinates z_1 and z_2. Then we would obtain $\lambda_w(\zeta_1, \zeta_2) = (1/2\pi) [q_0(\zeta_2) - q_0(\zeta_1)]$ where q_0 is harmonic on $R \sim \{\zeta_1, \zeta_2\}$; $q_0(w_j) + (-1)^j \log |w_j|$ is harmonic near $w_j = 0$ and $q_0 = L_0 q_0$ in a neighborhood of the ideal boundary of R.

The functions p_0 and q_0 differ by a constant. Let $z_j(w_j)$ denote the conformal mapping of a neighborhood of $w_j = 0$ onto a neighborhood of $z_j = 0$ which is induced by the local coordinates z_j and w_j. The function $p_0 - q_0$ becomes harmonic on R upon setting

$$(p_0 - q_0)(\zeta_1) = p_0(\zeta_1) - q_0(\zeta_1) + \log |z'_1(0)|,$$

$$(p_0 - q_0)(\zeta_2) = p_0(\zeta_2) - q_0(\zeta_2) - \log |z'_2(0)|.$$

Since $p_0 - q_0$ is now harmonic on R and has L_0-behavior in a neighborhood of the ideal boundary of R, $p_0 - q_0$ must be constant. In particular, its values at ζ_1 and ζ_2 coincide, which gives

$$\lambda_w(\zeta_1, \zeta_2) = \lambda_z(\zeta_1, \zeta_2) - (1/2\pi) \log |z'_1(0)z'_2(0)|.$$

This demonstrates that the definition of the reduced extremal distance $\lambda(\zeta_1, \zeta_2)$ is not a conformal invariant. It also shows that $\exp[-2\pi\lambda(z_1, z_2)]|dz_1| |dz_2|$ is an invariant form.

(4.4) Other reduced extremal distances between two points can be introduced by using the compactifications R' and R^* of R. Take $\mathcal{F}'(r_1, r_2)$ (respectively $\mathcal{F}^*(r_1, r_2)$) to be the collection of all curves in $R' \sim [D_1(r_1) \cup D_2(r_2)]$ ($R^* \sim [D_1(r_1) \cup D_2(r_2)]$) joining $\alpha_1(r_1)$ to $\alpha_2(r_2)$. $\lambda(\mathcal{F}'(r_1, r_2))$ (respectively $\lambda(\mathcal{F}^*(r_1, r_2))$) is the extremal length of the family of chains obtained by restricting each curve in $\mathcal{F}'(r_1, r_2)$ (respectively, $\mathcal{F}^*(r_1, r_2)$) to R. The limits

$$\lambda'(\zeta_1, \zeta_2) = \lim_{(r_1, r_2) \to (0,0)} [\lambda(\mathcal{F}'(r_1, r_2)) + (1/2\pi) \log(r_1 r_2)],$$

$$\lambda^*(\zeta_1, \zeta_2) = \lim_{(r_1, r_2) \to (0,0)} [\lambda(\mathcal{F}^*(r_1, r_2)) + (1/2\pi) \log(r_1 r_2)]$$

both exist and are always finite. The existence of the limits is established exactly as in Proposition 2. $\lambda'(\zeta_1, \zeta_2)$ is the Alexandroff one-point compactification reduced extremal distance between ζ_1 and ζ_2. $\lambda^*(\zeta_1, \zeta_2)$ is the reduced extremal distance between ζ_1 and ζ_2 relative to the Kerékjártó-Stoïlow compactification. Because $\mathcal{F}(r_1, r_2) \subset \mathcal{F}'(r_1, r_2), \mathcal{F}^*(r_1, r_2)$, it follows that $\lambda(\zeta_1, \zeta_2) \geq \lambda'(\zeta_1, \zeta_2), \lambda^*(\zeta_1, \zeta_2)$. Also, if R is compact, then $\mathcal{F}(r_1, r_2) = \mathcal{F}'(r_1, r_2) = \mathcal{F}^*(r_1, r_2)$ and $\lambda(\zeta_1, \zeta_2) = \lambda'(\zeta_1, \zeta_2) = \lambda^*(\zeta_1, \zeta_2)$.

Both $\lambda'(\zeta_1, \zeta_2)$ and $\lambda^*(\zeta_1, \zeta_2)$ are also related to principal functions. Let $p_I(\zeta) = p_I(\zeta; \zeta_1, \zeta_2)$ and $p_Q(\zeta) = p_Q(\zeta; \zeta_1, \zeta_2)$ be the $(I)L_1$ and $(Q)L_1$ principal functions, respectively, having the singularity $(-1)^{j+1} \log |z_j|$ at ζ_j. The values $p_I\langle\zeta_j\rangle$ and $p_Q\langle\zeta_j\rangle$ are defined in the same manner as $p_0\langle\zeta_j\rangle$. A proof of the following theorem would just parallel the proof of Theorem 5 except that $(I)L_1$ or $(Q)L_1$ behavior must be substituted whenever L_0 behavior occurs in the original proof. In the course of proving Theorem 5 we noted that the proof still went through under these conditions.

Theorem 6.

$$\lambda'(\zeta_1, \zeta_2) = (1/2\pi)[p_I\langle\zeta_2\rangle - p_I\langle\zeta_1\rangle],$$

$$\lambda^*(\zeta_1, \zeta_2) = (1/2\pi)[p_Q\langle\zeta_2\rangle - p_Q\langle\zeta_1\rangle].$$

5. **Pseudometrics obtained from classes of harmonic functions.** As usual R is an arbitrary Riemann surface. $HD(R)$ is the class of all harmonic functions on R which have finite Dirichlet integral $D(u)$. $HD_1(R) = \{u \in HD(R): D(u) \le 1\}$. $KD(R)$ consists of those $u \in HD(R)$ which have the property that $\int_\gamma *du = 0$ for all dividing cycles γ on R. When R is a region in the complex plane, every cycle is dividing and $KD(R)$ is the class of harmonic functions on R which possess a finite Dirichlet integral and have a single-valued harmonic conjugate in R. Set $KD_1(R) = KD(R) \cap HD_1(R)$. Both $HD(R)$ and $KD(R)$ are vector spaces under point-wise addition and scalar multiplication. If any two functions in $HD(R)$ which differ by a constant are identified, then $HD(R)$ becomes a Hilbert space with norm $\|u\| = D(u)^{1/2}$ and inner product $(u, v) = D(u, v)$, the mixed Dirichlet integral. The subspace $KD(R)$ is also a Hilbert space. The orthogonal complement of $KD(R)$ in $HD(R)$ is denoted by $HM(R)$ and is called the class of harmonic measures. Set $HM_1(R) = HM(R) \cap HD_1(R)$. (The orthogonal decomposition $HD(R) = HM(R) \oplus KD(R)$ of the space $HD(R)$ is really just the orthogonal decomposition $\Gamma_{he} = \Gamma_{hm} \oplus \Gamma_{he} \cap \Gamma_{hse}^*$ in the space of exact harmonic differentials [8, p. 194].) The classes $HD(R)$, $KD(R)$ and $HM(R)$ always contain the constant functions. In case one of these classes contain no additional functions, then we say R is of class O_{HD}, O_{KD} or O_{HM}, respectively.

There is a simple sufficient condition for a family of functions $\{u\}$ in $HD(R)$ to constitute a normal family. If $\{D(u)\}$ is uniformly bounded and for some point $\zeta_0 \in R$ the set $\{u(\zeta_0)\}$ is also uniformly bounded, then $\{u\}$ is a normal family [8, p. 37].

(5.1) First we define a pseudometric associated with $HD(R)$. For $\zeta_1, \zeta_2 \in R$ set

$$d_H(\zeta_1, \zeta_2) = d_{HD}(\zeta_1, \zeta_2) = \sup_{u \in HD_1(R)} |u(\zeta_1) - u(\zeta_2)|.$$

If $R \notin O_{HD}$, then it is sufficient to take the supremum over those $u \in HD(R)$ with $D(u) = 1$. The number $d_H(\zeta_1, \zeta_2)$ is the best possible constant such that

$$|u(\zeta_1) - u(\zeta_2)| \le d_H(\zeta_1, \zeta_2) D(u)^{1/2}$$

for all $u \in HD(R)$. The function d_H is defined on $R \times R$ and assumes nonnegative real values. It is not immediate that d_H is always finite. Notice that $d_H(\zeta_1, \zeta_2)$ $= \sup u(\zeta_1)$ where the supremum is taken over the compact normal family of all $u \in HD_1(R)$ with $u(\zeta_2) = 0$. If $HD_1(R)$ is given the topology of uniform convergence on compact subsets of R, then the functional $u \to u(\zeta_1)$ is continuous on this compact subfamily. Hence, the least upper bound of the functional is actually attained and d_H is finite-valued everywhere. The function d_H has several interesting properties.

Proposition 3. (i) d_H is a conformal invariant,

(ii) d_H is a pseudometric on R,

(iii) d_H is a monotone decreasing function of R,

(iv) the topology on R generated by d_H is coarser than the original topology,

(v) d_H is a continuous function on $R \times R$,

(vi) d_H is 2-subharmonic on $R \times R$; in other words, d_H is subharmonic as a function of each variable separately.

Proof. (i) The conformal invariance of d_H is a simple consequence of the fact that harmonicity and the Dirichlet integral are invariant under conformal mappings.

(ii) Clearly $\zeta_1 = \zeta_2$ implies $d_H(\zeta_1, \zeta_2) = 0$ and that d_H is symmetric is obvious. The triangle inequality follows from the definition of d_H and the ordinary triangle inequality.

(iii) Suppose that S is a Riemann surface which contains R and $\zeta_1, \zeta_2 \in R$. Consider $HD_1(S)$ as a subset of $HD_1(R)$ by restricting to R each function in $HD_1(S)$. Then $d_{H(S)}(\zeta_1, \zeta_2) \le d_{H(R)}(\zeta_1, \zeta_2)$ is immediate from the definition of d_H.

(iv) Given $\zeta_0 \in R$ and $\epsilon > 0$ we must show that there is a neighborhood N of ζ_0 with the property that $N \subset \{\zeta : d_H(\zeta_0, \zeta) < \epsilon\}$. Let V be a parametric disk centered at ζ_0 with local coordinate z. This means that $z(\zeta_0) = 0$ and $z(V) = U$ is the unit disk in the complex plane. Employing the monotonicity and conformal invariance of d_H, we obtain

$$d_{H(R)}(\zeta_0, \zeta_1) \le d_{H(V)}(\zeta_0, \zeta_1) = d_{H(U)}(0, z_1)$$

where $\zeta_1 \in V$ and $z_1 = z(\zeta_1) \in U$. It is possible to compute $d_{H(U)}(0, z_1)$ explicitly. Let $u \in HD(U)$, then there is a holomorphic function f defined in U the real part of which is u. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $D(u) = \pi \sum_{n=1}^{\infty} n|a_n|^2$. Using the Cauchy-Schwarz inequality, we have

$$|u(z_1) - u(0)|^2 \leq |f(z_1) - f(0)|^2 = \left| \sum_{n=1}^{\infty} a_n z_1^n \right|^2$$

$$\leq \sum_{n=1}^{\infty} n|a_n|^2 \sum_{n=1}^{\infty} \frac{|z_1|^2}{n} = \frac{1}{\pi} D(u) \log(1 - |z_1|^2)^{-1}$$

and equality holds if $u(z) = \log|1 - \bar{z}_1 z|^{-1}$. Therefore, $d_{H(U)}(0, z_1) = [(1/\pi) \log(1 - |z_1|^2)^{-1}]^{\frac{1}{2}}$. Select $\delta > 0$ so that $d_{H(U)}(0, z) < \epsilon$ for $|z| < \delta$. Then $N = \{\zeta : |z(\zeta)| < \delta\}$ is a suitable neighborhood of ζ_0.

(v) Since $d_H(\zeta_1, \zeta_2)$ is defined as the supremum of the family of continuous functions $|u(\zeta_1) - u(\zeta_2)|$, where $u \in HD_1(R)$, d_H is certainly lower semicontinuous on $R \times R$. Establishing upper semicontinuity requires a normal family argument. Let a be the upper limit of d_H at the point (ζ_1, ζ_2); we must show that $a \leq d_H(\zeta_1, \zeta_2)$. Let the sequence $\{\zeta_{jn}\}$ be such that $\zeta_{jn} \to \zeta_j$ and $d_H(\zeta_{1n}, \zeta_{2n}) \to a$. There is a function $u_n \in HD_1(R)$ with $u_n(\zeta_{1n}) = d_H(\zeta_{1n}, \zeta_{2n})$ and $u(\zeta_{2n}) = 0$. Since $\zeta_{2n} \to \zeta_2$ and the topology induced by d_H is coarser than the original topology, we may assume that $d_H(\zeta_2, \zeta_{2n}) < 1$ for all n. Then $|u_n(\zeta_2)| = |u_n(\zeta_2) - u_n(\zeta_{2n})| < 1$ for every n so that $\{u_n\}$ is a normal family. This permits us to suppose that $\{u_n\}$ is uniformly convergent on compact subsets of R to a function $u \in HD_1(R)$. Then $u(\zeta_2) = \lim u_n(\zeta_{2n}) = 0$ and $a = \lim_{n \to \infty} u_n(\zeta_{1n}) = u(\zeta_1) \leq d_H(\zeta_1, \zeta_2)$.

(vi) The supremum of a family of subharmonic functions is subharmonic provided the supremum is upper semicontinuous. For a fixed $\zeta_2 \in R$ the function $|u(\zeta) - u(\zeta_2)|$ is subharmonic on R. The supremum $d_H(\zeta, \zeta_2)$ of all such functions in $HD_1(R)$ is continuous and hence subharmonic. By symmetry $d_H(\zeta_1, \zeta_2)$ is also subharmonic as a function of ζ_2.

(5.3) In the same manner pseudometrics can be introduced on R through the classes $KD(R)$ and $HM(R)$. Define

$$d_K(\zeta_1, \zeta_2) = d_{KD}(\zeta_1, \zeta_2) = \sup_{u \in KD_1(R)} |u(\zeta_1) - u(\zeta_2)|,$$

$$d_M(\zeta_1, \zeta_2) = d_{HM}(\zeta_1, \zeta_2) = \sup_{u \in HM_1(R)} |u(\zeta_1) - u(\zeta_2)|.$$

Because $KD_1(R)$, $HM_1(R) \subset HD_1(R)$, it is evident that $d_K(\zeta_1, \zeta_2)$, $d_M(\zeta_1, \zeta_2) \leq d_H(\zeta_1, \zeta_2)$. Both d_K and d_M share the properties listed for d_H in Proposition 3.

(5.4) In general, d_H, d_K and d_M are just pseudometrics and not metrics. Clearly, $d_H(\zeta_1, \zeta_2) = 0$ if and only if $u(\zeta_1) = u(\zeta_2)$ for all $u \in HD(R)$. Similar statements hold for d_K and d_M. Both d_H and d_K are metrics when R is a region in the complex plane having finite area $A(R)$. In this situation both harmonic functions $A(R)^{-\frac{1}{2}}x$ and $A(R)^{-\frac{1}{2}}y$ belong to $KD_1(R) \subset HD_1(R)$. When $z_1 = x_1 + iy_1 \neq z_2 = x_2 + iy_2$, it follows that

$$d_H(z_1, z_2) \geq d_K(z_1, z_2) \geq A(R)^{-\frac{1}{2}} \max\{|x_1 - x_2|, |y_1 - y_2|\} > 0.$$

For a general region R in the extended complex plane d_K is a metric if and only if $R \notin O_{KD}$ [1, Theorem 7, p. 113]. Thus, either d_K is a metric or $d_K \equiv 0$ for subregions of the extended complex plane. Whether a similar result holds for d_H is not known. Rodin [7] showed that for a nonplanar Riemann surface d_H need not be a metric even though $R \notin O_{HD}$; the surface which he constructs has infinite genus. When R is a simply connected Riemann surface $d_M \equiv 0$.

(5.5) The fact that d_H is always finite shows that $u \to u(\zeta_2) - u(\zeta_1)$ is a bounded linear functional on the Hilbert space $HD(R)$. The Riesz representation theorem guarantees that there is a function $q_H \in HD(R)$, unique up to an additive constant, such that $u(\zeta_2) - u(\zeta_1) = D(u, q_H)$ for all $u \in HD(R)$. Clearly, q_H depends upon the points ζ_1, ζ_2. Schwarz' inequality produces $|u(\zeta_1) - u(\zeta_2)|^2 \leq D(u)D(q_H)$ with equality if and only if $u = aq_H + b$ where a and b are constants. This proves that

(5.5.1)
$$d_H(\zeta_1, \zeta_2)^2 = D(q_H) = q_H(\zeta_2) - q_H(\zeta_1).$$

It is known that $q_H = (1/2\pi)(p_0 - p_I)$ [9, Theorem 2H, p. 39].

In the same manner, $d_K(\zeta_1, \zeta_2)^2 = D(q_K) = q_K(\zeta_2) - q_K(\zeta_1)$ where $q_K = (1/2\pi)(p_0 - p_Q) \in KD(R)$ [9, Theorem 2H, p. 39]. Also, $d_M(\zeta_1, \zeta_2)^2 = D(q_M) = q_M(\zeta_2) - q_M(\zeta_1)$ where $q_M = (1/2\pi)(p_Q - p_I) \in HM(R)$ [8, p. 103]. Notice that $q_H = q_K + q_M$ is just the orthogonal decomposition of $q_H \in HD(R)$ relative to the orthogonal complements $KD(R)$ and $HM(R)$.

(5.6) Now we come to our main result which expresses the pseudometrics d_H, d_K and d_M in terms of the reduced extremal distances λ, λ' and λ^*.

Theorem 7.
$$d_H(\zeta_1, \zeta_2)^2 = \lambda(\zeta_1, \zeta_2) - \lambda'(\zeta_1, \zeta_2),$$
$$d_K(\zeta_1, \zeta_2)^2 = \lambda(\zeta_1, \zeta_2) - \lambda^*(\zeta_1, \zeta_2),$$
$$d_M(\zeta_1, \zeta_2)^2 = \lambda^*(\zeta_1, \zeta_2) - \lambda'(\zeta_1, \zeta_2),$$

where λ, λ' and λ^* are all computed with respect to the same local coordinates at ζ_1 and ζ_2.

Proof. Observe that $q_H(\zeta_j) = (1/2\pi)(p_0\langle\zeta_j\rangle - p_I\langle\zeta_j\rangle)$ $(j = 1, 2)$. Then by Theorems 5 and 6 and (5.5.1) we obtain $d_H(\zeta_1, \zeta_2)^2 = \lambda(\zeta_1, \zeta_2) - \lambda'(\zeta_1, \zeta_2)$. The results for d_K and d_M are established in the same manner.

Remark. Although the reduced extremal distances λ, λ' and λ^* are not conformal invariants, the differences $\lambda - \lambda'$, $\lambda - \lambda^*$ and $\lambda^* - \lambda'$ are.

6. **Applications.** Using the preceding results we can obtain new extremal length tests to decide whether a Riemann surface is of class O_{HD} or O_{HM}. These tests are analogs of a theorem of Rodin dealing with O_{KD} [6], [8, Theorem 2A, p. 206]. From the definition of d_H it is clear that $R \in O_{HD}$ if and only if $d_H \equiv 0$.

By Theorem 7 a necessary and sufficient condition in order that d_H vanish identically is for the two reduced extremal distances $\lambda(\zeta_1, \zeta_2)$ and $\lambda'(\zeta_1, \zeta_2)$ to be equal for all $\zeta_1, \zeta_2 \in R$. Of course, the reduced extremal distances must be computed relative to the same local coordinates at both ζ_1 and ζ_2. Similarly, $R \in O_{KD}$ (respectively, O_{HM}) if and only if $\lambda(\zeta_1, \zeta_2) = \lambda^*(\zeta_1, \zeta_2)$ (respectively, $\lambda'(\zeta_1, \zeta_2) = \lambda^*(\zeta_1, \zeta_2)$) for all $\zeta_1, \zeta_2 \in R$. These simple observations form the bases for the various tests.

(6.1) Suppose that E_0 and E_1 are compact subsets of a Riemann surface R. Let $\mathcal{F}(E_0, E_1)$ be the family of all curves $c: [0, 1] \to R$ such that $c(0, 1) \subset R \sim (E_0 \cup E_1)$ and $c(j) \in E_j$ $(j = 0, 1)$. Less precisely, c is a curve in $R \sim (E_0 \cup E_1)$ joining E_0 to E_1. $\lambda(\mathcal{F}(E_0, E_1))$ is the extremal distance of E_0, E_1 and is denoted by $\lambda(E_0, E_1)$. The same extremal length would be obtained by considering all curves on R with one endpoint in E_0 and the other in E_1. Recall that R' and R^* denote the Alexandroff one-point and Kerékjártó-Stoïlow compactification of R, respectively. Let $\mathcal{F}'(E_0, E_1)$ $(\mathcal{F}^*(E_0, E_1))$ be the family of all curves in $R' \sim (E_0 \cup E_1)$ $(R^* \sim (E_0 \cup E_1))$ joining E_0 to E_1. $\lambda'(E_0, E_1)$ $(\lambda^*(E_0, E_1))$ is the extremal length of the family of chains obtained by restricting each curve in $\mathcal{F}'(E_0, E_1)$ $(\mathcal{F}^*(E_0, E_1))$ to R.

(6.2) Rodin established the following result [6].

Theorem 8. *A necessary and sufficient condition that* $\lambda(E_0, E_1) = \lambda^*(E_0, E_1)$ *for all pairs of compact sets* E_0, E_1 *of* R *is that* $R \in O_{KD}$.

This theorem has analogs for both O_{HD} and O_{HM}.

Theorem 9. *A necessary and sufficient condition that* $\lambda(E_0, E_1) = \lambda'(E_0, E_1)$ *(respectively,* $\lambda'(E_0, E_1) = \lambda^*(E_0, E_1)$*) for all pairs of compact sets* E_0, E_1 *of* R *is that* $R \in O_{HD}$ *(respectively,* O_{HM}*).*

Proof. The proof of sufficiency is an adaptation of Rodin's, while the necessity proof is new and also yields a simpler proof in the case of his result. Thus, these three tests for O_{HD}, O_{KD} and O_{HM} may be derived in a uniform manner. We shall establish the result in detail only for the O_{HD} test since the case dealing with O_{HM} just parallels the given proof.

To prove the necessity, assume that the extremal distances $\lambda(E_0, E_1), \lambda'(E_0, E_1)$ are always equal. Take distinct points $\zeta_1, \zeta_2 \in R$; it suffices to show that $\lambda(\zeta_1, \zeta_2) = \lambda'(\zeta_1, \zeta_2)$. Let z_j be a local parameter defined in a neighborhood of ζ_j with $z_j(\zeta_j) = 0$. For r positive and small, $E_0 = \text{Cl}(D_1(r))$ and $E_1 = \text{Cl}(D_2(r))$ are disjoint compact subsets of R. Note that $\mathcal{F}(E_0, E_1) = \mathcal{F}(r, r)$ and $\mathcal{F}'(E_0, E_1) = \mathcal{F}'(r, r)$ in this case. Consequently, $\lambda(\mathcal{F}(r, r)) = \lambda(\mathcal{F}'(r, r))$ so that $\lambda(\zeta_1, \zeta_2) = \lambda'(\zeta_1, \zeta_2)$.

Similarly, $\lambda'(E_0, E_1) = \lambda^*(E_0, E_1)$ implies that $\lambda'(\zeta_1, \zeta_2) = \lambda^*(\zeta_1, \zeta_2)$ and $R \in O_{HM}$. Also, if $\lambda(E_0, E_1) = \lambda^*(E_0, E_1)$ for all compact $E_0, E_1 \subset R$, then $\lambda(\zeta_1, \zeta_2) = \lambda^*(\zeta_1, \zeta_2)$ and $R \in O_{KD}$. This is the short new proof of half of Rodin's result.

The sufficiency is more complicated. Suppose that $R \in O_{HD}$ and let E_0, E_1 be compact subsets of R. If $E_0 \cap E_1 \neq \emptyset$, then $\lambda(E_0, E_1) = \lambda'(E_0, E_1) = \infty$. Therefore we may assume that E_0 and E_1 are disjoint. First we deal with the simple situation in which $R \sim ((E_0 \cup E_1)^0) = \overline{W}$ is a bordered Riemann surface having compact border $\partial E_0 \cup \partial E_1$. Then by Theorems 2 and 3, $\lambda(E_0, E_1) = D_W(u_0)^{-1}$ and $\lambda'(E_0, E_1) = D_W(u_I)^{-1}$ where u_0, u_I are harmonic on $\overline{W}$, $u_0 = u_I = 0$ on ∂E_0, $u_0 = u_I = 1$ on ∂E_1, and $u_0 = L_0 u_0$, $u_I = (I)L_1 u_I$ in a neighborhood of the ideal boundary of $\overline{W}$ (which is also a neighborhood of the ideal boundary of R). But $(I)L_1 = L_0$ because $R \in O_{HD}$ [8, Theorem 1A, p. 201]. Therefore, $u_0 = u_I$ and $\lambda(E_0, E_1) = \lambda'(E_0, E_1)$.

If $\lambda^*(E_0, E_1)$ were being considered in this situation, then, from Theorem 4, $\lambda^*(E_0, E_1) = D_W(u_Q)^{-1}$ where u_Q is harmonic on $\overline{W}$, $u_Q = 0$ on ∂E_0, $u_Q = 1$ on ∂E_1, and $u_Q = (Q)L_1 u_Q$ in a neighborhood of the ideal boundary of $\overline{W}$. Rodin's result here uses the fact that $R \in O_{KD}$ if and only if $L_0 = (Q)L_1$ [8, Theorem 1B, p. 201]. The case dealing with O_{HM} uses the result that $R \in O_{HM}$ if and only if $(I)L_1 = (Q)L_1$ [8, Theorem 1C, p. 202].

Next, we establish the result when $R \sim (E_0 \cup E_1)$ is connected. In this situation $R \sim (E_0 \cup E_1)$ is a Riemann surface and may be expressed as an increasing union of compact bordered Riemann surfaces. Let $\{\overline{R}_n\}$ be an increasing sequence of bordered Riemann surfaces having compact border $\partial \overline{R}_n$ such that (i) $R \sim (E_0 \cup E_1) \supset \overline{R}_n$ for all n, (ii) $R = \bigcup \overline{R}_n$ and (iii) each component of $R \sim \overline{R}_n$ meets one of E_0 and E_1 but not both. Divide the contours of $\partial \overline{R}_n$ into two disjoint sets α_{0n} and α_{1n} according as the contours bound components of $R \sim \overline{R}_n$ which meet E_0 or E_1. The first portion of the proof guarantees that $\lambda(\alpha_{0n}, \alpha_{1n}) = \lambda'(\alpha_{0n}, \alpha_{1n})$. Now, by using the extremal length technique originated by Beurling, it can be shown that $\lambda(\alpha_{0n}, \alpha_{1n}) \rightarrow \lambda(E_0, E_1)$ and $\lambda'(\alpha_{0n}, \alpha_{1n}) \rightarrow \lambda'(E_0, E_1)$. Hence, $\lambda(E_0, E_1) = \lambda'(E_0, E_1)$ in this situation.

Finally, the general case is handled by applying this last result to each component of $R \sim (E_0 \cup E_1)$ and then using the fact that the reciprocal of extremal length is additive over families of curves having disjoint supports [5, Theorem 2.9, p. 78].

Corollary. $HD(R) = KD(R)$ *if and only if* $\lambda'(E_0, E_1) = \lambda^*(E_0, E_1)$ *for all pairs of compact sets* E_0, E_1 *in* R.

Proof. $HD(R) = KD(R)$ if and only if $R \in O_{HM}$ [8, Theorem 1C, p. 202].

Given a compact set $E \subset R$ let $\lambda(E, \beta)$ denote the extremal length of the family $\mathcal{F}'(E)$ of curves $c: [0, 1) \rightarrow R$ such that $c(0) \in E$ and $c(t) \rightarrow \beta$ as $t \rightarrow 1$ where β is the point added to R to form R'.

Corollary. *If* $\lambda(E, \beta) = \infty$ *for every compact set* $E \subset R$, *then* $R \in O_{HD}$.

Proof. Let E_0, E_1 be compact subsets of R. It suffices to show that $\lambda(\mathcal{F}) = \infty$ where $\mathcal{F} = \mathcal{F}'(E_0, E_1) \sim \mathcal{F}(E_0, E_1)$ since then $\lambda(E_0, E_1) = \lambda'(E_0, E_1)$. Because every curve in $\mathcal{F}$ contains a subcurve in $\mathcal{F}'(E_j)$ $(j = 0, 1)$, it follows that $\lambda(\mathcal{F}) \geq \lambda(E_j, \beta) = \infty$.

Let O_G be the class of Riemann surfaces which do not possess a Green's function. $R \in O_G$ if and only if $\lambda'(E, \beta) = \infty$ for all compact sets $E \subset R$ [8, Remark 1, p. 200]. For a planar Riemann surface $R \in O_G$ if and only if $R \in O_{HD}$ [8, Theorem IF, p. 203]. In general, $O_G \subsetneq O_{HD}$ [3, pp. 256–264].

7. **Examples.** Explicit computations may be given in the case of a hyperbolic simply connected Riemann surface.

(7.1) First consider the unit disk U. Then $HD(U) = KD(U)$, $p_I = p_Q$ and $\lambda' = \lambda^*$. Given $z_1, z_2 \in U$,

$$p_I(z; z_1, z_2) = g(z, z_2) - g(z, z_1) = \log \left| \frac{(z - z_1)(1 - \bar{z}_2 z)}{(z - z_2)(1 - \bar{z}_1 z)} \right|,$$

$$p_0(z; z_1, z_2) = N(z, z_2) - N(z, z_1) = \log \left| \frac{(z - z_1)(1 - \bar{z}_1 z)}{(\bar{z} - z_2)(1 - \bar{z}_2 z)} \right|,$$

where $g(z, z_j)$ and $N(z, z_j)$ are Green's and Neumann's function, respectively, for U having logarithmic singularity at z_j. Hence,

$$q_H(z; z_1, z_2) = \frac{1}{\pi} \log \left| \frac{1 - \bar{z}_1 z}{1 - \bar{z}_2 z} \right|.$$

If $z - z_j$ is used as a local coordinate at z_j, then we obtain

$$\lambda(z_1, z_2) = \frac{1}{2\pi} \log \frac{|z_1 - z_2|^2 |1 - \bar{z}_1 z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)},$$

$$\lambda'(z_1, z_2) = \frac{1}{2\pi} \log \frac{|z_1 - z_2|^2 (1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \bar{z}_1 z_2|^2},$$

$$d_H(z_1, z_2)^2 = \frac{1}{\pi} \log \frac{|1 - \bar{z}_1 z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}.$$

These results are easily transformed to an arbitrary hyperbolic simply connected region in the complex plane. Let $z = f(w)$ map R conformally onto U. Then $d_{H(R)}(w_1, w_2) = d_{H(U)}(f(w_1), f(w_2))$ and, if $w - w_j$ is used as a local coordinate at w_j, then

$$\lambda_R(w_1,\, w_2) = \lambda_U(f(w_1),\, f(w_2)) - (1/2\pi) \log |f'(w_1) f'(w_2)|$$

with a similar result holding for $\lambda_R'(z_1, z_2)$. If R is a Riemann surface and $z = f(w)$ maps R conformally onto U, then $\lambda_R(w_1, w_2) = \lambda_U(f(w_1), f(w_2))$ if $f(w) - f(w_j)$ is used as a local coordinate at w_j.

(7.2) In a hyperbolic simply connected region R, d_H is a conformally invariant metric and may be compared with the Poincaré hyperbolic metric d. The following results are readily established for U:

(i) $\qquad d_H(z_1, z_2)^2 = (2/\pi) \log \cosh d(z_1, z_2),$

(ii) $\qquad d(z_1, z_2) \geq \pi^{1/2} d_H(z_1, z_2)$ and equality holds if and only if $z_1 = z_2$,

(iii) $\qquad \lim_{z \to z_0} \dfrac{\pi^{1/2} d_H(z, z_0)}{|z - z_0|} = \dfrac{1}{1 - |z_0|^2} = \lim_{z \to z_0} \dfrac{d(z, z_0)}{|z - z_0|}.$

If R is an arbitrary Riemann surface and $R \notin O_{HD}$, then U is the universal covering surface for R so that the hyperbolic metric d may be introduced on R. The simple relationship (i) between d and d_H no longer holds on R. For instance, let $R = \{z: 0 < |z| < 1\}$, then $HD(R) = HD(U)$ because $z = 0$ is a removable singularity for every function in $HD(R)$. Therefore, $d_{H(U)} = d_{H(R)}$, but for any fixed $z_0 \in R$, the hyperbolic distance $d(z, z_0) \to \infty$ as $z \to 0$ (d is the hyperbolic metric on R not in U).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92037

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address: Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221